Linear Algebra 1

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Linear Algebra is the study of linear maps.

What makes a map 'linear'? Suppose c_1 and c_2 are constants (real constants, lets say 5 and 11) and **v** and **w** are in the domain of the function. Then the function f is **linear** if $f(c_1v + c_2w) = c_1f(v) + c_2f(w)$.

That is if we can take the constant 'common' and compute the effect of the function separately on the input v and w, then we say that **f is linear.**

- The squaring function is **not linear** since $(a + b)^2 \neq a^2 + b^2$ in most cases.
- Other functions which are **not linear** includes exponentiation, logarithm, trigonometric functions, factorial etc.

But there are a ton of functions which **are linear**.

- Differentiation
- Rotation
- Definite Integration
- Dilation

For example suppose f and g be two differentiable functions. Then we know that $\frac{d}{dx}\left(f(x)+f(x)\right)=\frac{d}{dx}\left(f(x)\right)+\frac{d}{dx}((g(x))$

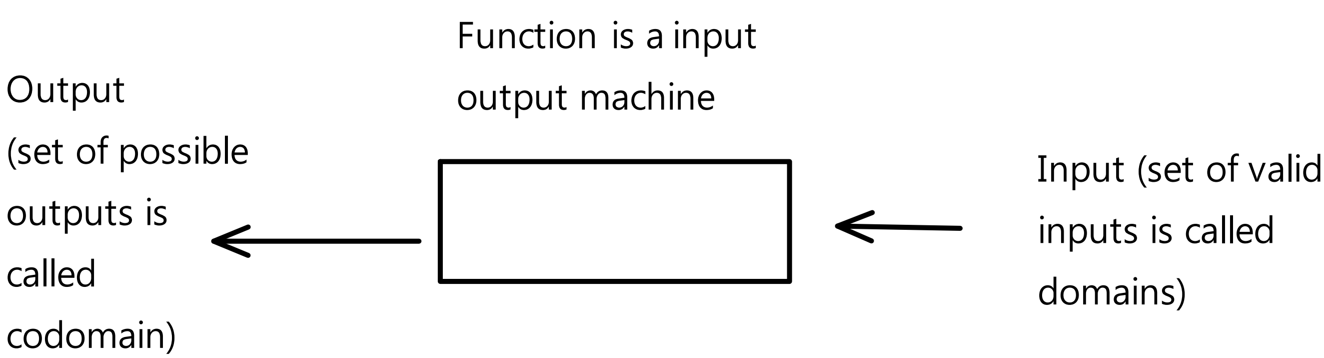
Note that here 'taking derivative' is the 'linear map' and 'differentiable functions' are inputs.

Proving a function to be linear can be a daunting task.

1.1 Show that differentiation is linear.

1.2 Show that rotation (of point in R^2) is linear.

Since we are studying linear maps, it makes sense to discuss the domains (valid inputs) and codomains (possible outputs) of such functions



Usually there is a difference between **codomain (possible outputs)** and **range (set of actual outputs).**

Example: Consider **squaring** . Then the **domain** consists of all real numbers. Suppose the **co-domain** is also set of all real numbers (it depends on us what we want the codomain to be). Then **range** is set of all non negative real numbers (as the squaring machine will give only non-negative outputs).

Note that by definition $range \subset codomain$

Since we are interested in linear maps, we will study their domains and co-domains.

Domain sets and Codomain sets of Linear maps have a special name: Vector Space

A Vector Space is actually a pair of interactive sets:

- Set of constant = Scalers (usually real numbers)
- Set of actual inputs = Vectors

What are the suitable candidates for scaler sets and vector sets?

For our purpose scalers are always real numbers. They are constants that get multiplied to the vectors.

On the other hand set of **vectors** can be quite diverse:

On the other hand set of **vectors** can be quite diverse.

- Differentiable Functions
- Continuous Function
- Polynomials of degree up to n
- Vectors from high school geometry (arrows connecting two points in the x-y plane).
- All matrices with determinant 1.

Possible Confusion: *In the context of linear algebra, the word **vector** has a very diverse scope. Do not confuse it with arrows from high school geometry. A vector, in linear algebra, can very well be a function, a polynomial, a matrix etc.*

What makes a set, 'vector set'?

** It must be **closed under linear combination** and **it must contain its own 0 (additive identity).**

(definition of vector space is quite long, but for our purpose ** is sufficient).

- A set V is said to be **closed** under linear combination: if $v, w \in V, then\ c_1 v + c_2 w \in V$. Here c_1, c_2 are scalers (real constants). Note that if $c_1 v + c_2 w$ is not inside V, then we cannot apply the linear map to it (as it is no longer inside the apply the linear map to it (as it is no longer inside the domain). Hence our study of linear maps is jeopardized. This is why we **require** the linear combination of domain elements to be inside domain.

- **Additive identity** is a fancy name for 0. Actually 0 can be *viewed* as a zero function, zero polynomial, zero matrix or the point (0,0) in x-y plane, depending on the context of the set of vectors. Hence it is given this name: additive identity. (*an element **e** is called additive identity if v + e = v for all vectors v in the domain).*

From now on, we will mention only the set of vectors, while talking about a vector space. Set of scalers should be understood to be 'there' and to be 'set of real numbers' unless otherwise mentioned.

1.3 Prove that the set of polynomials of degree 3, is not a vector space.

1.4 Prove that the set of polynomials **upto** degree 3 is a vector space.

1.5 Prove the parallelogram law of vector addition: $(x_1,y_1) + (x_2,y_2) = (x_1+x_2,y_1+y_2)$. If $A = (x_1,y_1), B =$

$(x_2, y_2), C = (x_1 + x_2, y_1 + y_2)$ then OABC form a parallelogram.