An Inaccessible Group

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1. Introduction

Stallings [6] showed that a group G has more than one end if and only if $G \approx A *_F B$, where F is finite, $A \neq F \neq B$, or G is an HNN-extension with finite edge group F.
A finitely generated group G is said to be *accessible* if it is the fundamental group of a graph of groups in which all edge groups are finite and every vertex group has at most one end. We say that G is *inaccessible* if it is not accessible.

Let $d(G)$ denote the minimal number of generators of the finitely generated group G. It follows from Grushko's Theorem that $d(G*H) = d(G)+d(H)$. It follows that G is a free product of indecomposable groups, i.e. groups which cannot be written as a non-trivial free product. The problem of accessibility is whether we can replace the free product with free product with finite amalgamation in the last statement. (The number of HNN-decompositions is bounded by $d(G)$.) However, there is no analogue of Grushko's Theorem. In fact, if G is accessible then any process of sucessively decomposing G, and the factors that arise in the process, terminates after a finite number of steps. See [2] for a proof of this and related results.
Linnell [5] proved that if G is finitely generated then, for any reduced decomposition of G as a graph of groups X in which all edge groups are finite, there is a bound B such that $\sum_{e \in E} 1/|G_e| < B$, where E is the edge set of X. Thus for any $k > 0$, there are at most kB edges e such that $|G_e| \leq k$. In [3] I showed that G is accessible if G is almost finitely presented. Groves and Swarup [4] have extended this result to a somewhat larger class of groups. This paper contains the construction of a finitely generated inaccessible group. C.T.C. Wall [8] conjectured that all finitely generated groups are accessible. On the other hand, Bestvina and Feighn [1] have given an example of a finitely generated group which does not satisfy a generalized accessibility condition

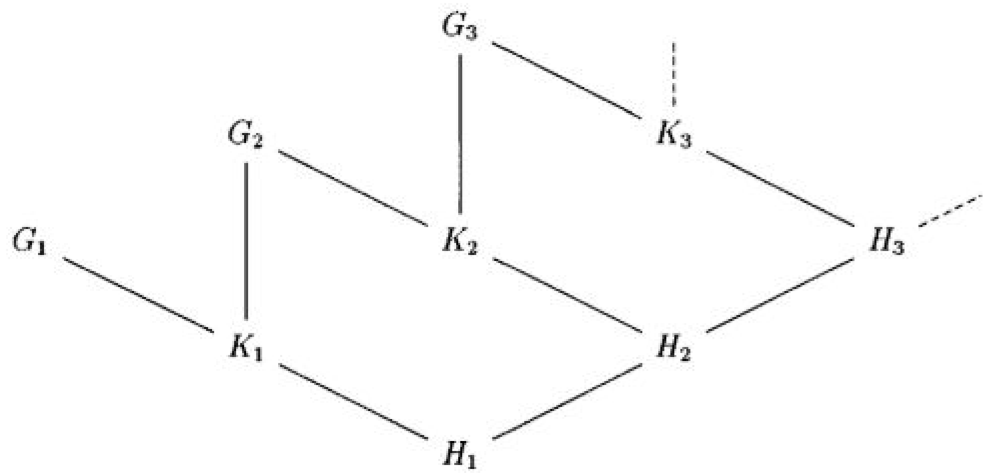
in which decompositions over torus subgroups are allowed. It was by thinking about their construction that I thought of the example presented here.

Let X be a connected locally finite graph. Thomassen and Woess [7] have defined X to be accessible if for some positive integer n any pair of ends of X can be separated by removing at most n edges. They show, by using results from [2] Chapter 2, that a finitely generated group G is accessible as a group if and only if its Cayley graph (with respect to a finite generating set) is accessible as a graph. They investigate alternative definitions for a graph to be accessible.

I am very grateful to Warren Dicks, Peter Kropholler and Martin Roller for providing short proofs that the group J is inaccessible to replace my laboured argument.

2. Constructing the example

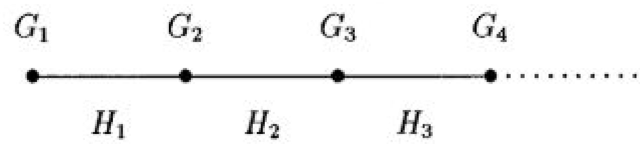
Suppose we have a lattice of groups as shown.



In the diagram lines represent proper inclusions. We also require that G_{i+1} is generated by K_i and H_{i+1}.

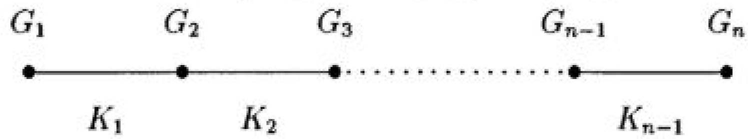
We show how to associate an inaccessible group with such a group lattice, when K_i (and hence H_i) is finite for all i, and G_1 is finitely generated. In the next section we show that such a lattice of groups exists.

Let P be the fundamental group of the graph of groups

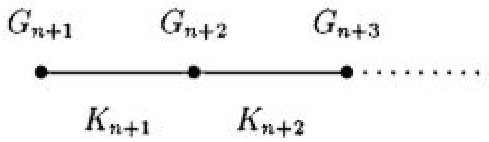


In P, we have $H_1 < H_2 < \ldots < H_\omega := \bigcup_{i \in \mathbb{N}} H_i$. Since H_ω is countable it can be embedded in a finitely generated group H. Let J be the free product with amalgamation $P *_{H_\omega} H$. Now J is generated by G_1 and H. For suppose L is the subgroup generated by G_1 and H. It suffices to show that $G_i < L$ for all $i \in \mathbb{N}$. But if $G_i < L$, then $K_i < L$ and $H_{i+1} < L$ and so $G_{i+1} < L$. It follows by induction that $G_i < L$ for all $i \in \mathbb{N}$. Hence J is finitely generated.

Let P_n be the fundamental group of the graph of groups



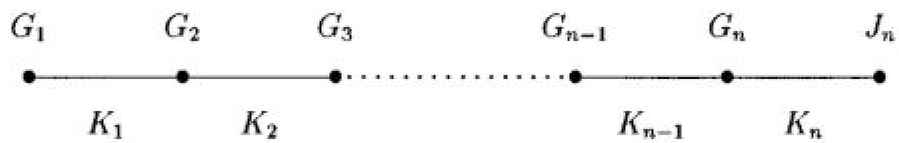
and let Q_n be the fundamental group of the graph of groups



Thus $P = P_n *_{K_n} Q_n$. Since $H_\omega < Q_n$, J decomposes as

$$J = P_n \underset{K_n}{*} (Q_n \underset{H_\omega}{*} H),$$

and so if $J_n = Q_n *_{H_\omega} H$, J decomposes as the fundamental group of the graph of groups



It follows immediately that J is inaccessible.

3. Constructing the lattice

In this section we construct a lattice of groups as specified in the previous section. Let H be the subgroup of $\mathrm{Symm}(\mathbb{Z})$ generated by the transposition $t = (0,1)$ and the shift map s, where $s(i) = i+1$. Put $t_i = s^i t s^{-i} = (i, i+1)$. Let $H_i = \langle t_{-i}, t_{-i+1}, \ldots, t_0, t_1, \ldots, t_{i-1} \rangle$. Thus H_i is isomorphic to the symmetric group S_{2i+1}. Let V be the group of all maps $\mathbb{Z} \to \mathbb{Z}_2$ with finite support, under the usual addition. Then H acts on V by $vh(n) = v(h(n))$ for all $v \in V$, $h \in H$ and $n \in \mathbb{Z}$. Let V_i be the subgroup of V consisting of all maps with support $[-i, i] = \{-i, -i+1, \ldots, 0, 1, \ldots, i\}$. Let $G_i' = V_i \rtimes H_i$. Let $z_i \in V_i$ with $z_i(n) = 1$ for $n \in [-i, i]$, then z_i is central in G_i'. Let $K_i = \langle z_i, H_i \rangle$ and note that $K_i = \mathbb{Z}_2 \times H_i$. For $i = 1, 2, \ldots$ let G_i be an isomorphic copy of G_i' and identify K_i with its image in G_i. We can then assume that $K_i = G_i \cap G_{i+1}$. It is left to the reader to check that the lattice of groups is as required.

References

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